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Structure of Weyl type Lie algebras ☆

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Abstract

Let A be a unital commutative associative algebra over a field F of characteristic zero, D a commutative subalgebra of $\mathrm{Der}_F(A)$ (all derivations of the associative algebra A). We assume that A is D-simple and denote the center of the Weyl type algebra $A[D]$ by F_1 which is an extension field of F when $A[D]$ is simple. In this paper, it is proved that the simple associative algebras $A[D]$ are noncommutative domains, and then the derivations of the simple associative algebras $A[D]$ and of the associated Lie algebras $A[D]^L$ are completely determined when $\dim_{F_1} F_1D < \infty$.

Keywords: Weyl type Lie algebras; Derivations

1. Introduction

Weyl algebras are a class of important simple associative algebras. The nth Weyl algebra A_n is the unital associative algebra over a field F with generators t_i, ∂_i for $i = 1, 2, \ldots, n$, satisfying the defining relations

$$[t_i, t_j] = 0 = [\partial_i, \partial_j], \tag{1.1}$$

$$[\partial_i, t_j] = \delta_{i,j} 1, \tag{1.2}$$

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for $1 \leqslant i, j \leqslant n$, where $[a, b] = ab - ba$ and $\delta_{i,j}$ is the Kronecker delta. There are many studies on Weyl algebras. In particular, irreducible weight representations of Weyl algebras were recently classified in [BBF].

Generalized Weyl algebras were introduced in [B1,B2], and studied in many references. Very recently, another generalization for Weyl algebras (called Weyl type algebras) was introduced in [SZ1,Z2]. The second author defined in [Z2] the Weyl type algebra $A[D]$ as follows (for more details, see Section 2). Suppose that A is a unital commutative associative algebra over a field F, and D an abelian subalgebra of $\mathrm{Der}(A)$ (all derivations of the associative algebra A). Then $A[D]$ is the polynomial algebra of D (as variables) with coefficients in A. Elements in this algebra act naturally on A as linear operators.

The simplicity and the structure of this Weyl type algebra $A[D]$ as an associative algebra and as a Lie algebra were studied in [Z2]. In [SZ2], isomorphisms between Weyl type (associative and Lie) algebras $A[D]$ and automorphisms are determined in the case when the action of D on A is locally finite. Derivations of the Lie algebras $A[D]$ were investigated in [Z1,LX,S] for different cases. Prior to the present paper, the best result on determining derivations of $A[D]$ belongs to [S] where Su assumed that D consists of locally finite derivations of A, D is finite dimensional, and F is algebraically closed. In our this paper we shall remove all these restrictions.

This paper is arranged as follows.

In Section 2, we recall some related concepts and results from [Br] and [Z2] and then deduce that the simple associative algebra $A[D]$ and the associated Lie algebra $A[D]^L$ have the same derivations (Theorem 2.4). In Section 3, we prove that if the associative algebra $A[D]$ is simple then it is a noncommutative domain (Theorem 3.1). We think this is a very interesting result of its own.

In Section 4, we introduce extended Weyl type algebras $A[X_1, \ldots, X_s][D]$ for a given Weyl type algebra $A[D]$ by a sequence $(f_1, f_2, \ldots, f_s) \in A^s$, and investigate some of their properties.

In Section 5, we assume that $A[D]$ is simple and $\dim_{F_1} F_1 D < \infty$, where F_1 is the center of $A[D]$ which is an extension filed of the base field F. In this case, D is not necessarily finite dimensional over F. We define two classes of derivations for $A[D]$, one is obtained by extending derivations of A and the other is obtained by restricting derivations of some extended Weyl type algebras $A[X_1, \ldots, X_s][D]$. Using the above established results and creating a powerful tool (push-up and pull-back technique), we completely determine all derivations of the simple associative algebras $A[D]$ and of the associated Lie algebras $A[D]^L$ (Theorem 5.3). More precisely, any derivation is a sum of an inner derivation and derivations from the above two classes. Actually we regard a derivation of $A[D]$ as a derivation from $A[D]$ to $A[X_1, \ldots, X_s][D]$, then determine all derivations from $A[D]$ to $A[X_1, \ldots, X_s][D]$, to obtain $\mathrm{Der}(A[D])$.

We hope the push-up-and-pull-back technique can be applied to some other similar situations of problems.

In the final section, Section 6, we provide several examples to show the power of our theorem. In particular, using our theorem we easily deduce all other known results on $\mathrm{Der}(A[D])$.

2. Definitions and preliminary results

Throughout this paper, we assume that F is a field of characteristic 0.

Let A be an associative algebra over F and B an associative subalgebra of A. An F-linear transformation $\partial : B \to A$ is called a derivation from B to A if

$$\partial(ab) = \partial(a)b + a\partial(b), \quad \forall a, b \in B.$$

Denote by $\mathrm{Der}_F(B, A)$ the F-vector space of all derivations from B to A. When $A = B$, simply write $\mathrm{Der}_F(A, A)$ as $\mathrm{Der}_F(A)$, and it is clear that $\mathrm{Der}_F(A)$ is a Lie algebra. Denote by A^L the associated Lie algebra of the associative algebra A. Denote by $\mathrm{Der}_F(A^L)$ the F-vector space of all derivations of the Lie algebra A^L.

Following [Z2], now we recall the definition for the Weyl type algebra $A[D]$. Let A be a unital commutative associative algebra over F, D a nonzero abelian subalgebra of $\mathrm{Der}_F(A)$. Let

$$\{\partial_i \mid i \in \dot{I}\}, \quad \text{where } \dot{I} \text{ is some indexing set},$$

be an F-basis of D. Denote

$$\dot{J} = \left\{\alpha = (\alpha_i \mid i \in \dot{I}) \mid \alpha_i \in \mathbb{Z}_+,\ \forall i \in \dot{I} \text{ and } \alpha_i = 0 \text{ for all but a finite number of } i \in \dot{I}\right\}.$$

For simplicities we write $\alpha = (\alpha_i \mid i \in \dot{I})$ as $\alpha = (\alpha_i)$ if there is no ambiguity. Define

$$|\alpha| = \sum_{i \in \dot{I}} \alpha_i,$$

which is called the *level* of α. For any $\alpha \in \dot{J}$, $x \in A$, we define the operator on A:

$$x\partial^\alpha = x \prod_{i \in \dot{I}} \partial_i^{\alpha_i},$$

by $(x\partial_1\partial_2 \cdots \partial_n)(y) = x(\partial_1(\partial_2 \cdots (\partial_n(y)) \cdots))$ for any $y \in A$. Let $A[D]$ be the F-vector space spanned by all these operators on A

$$x\partial^\alpha, \quad \forall x \in A;\ \alpha \in \dot{J}.$$

Note that the above elements can be linearly dependent. The associative product of $A[D]$ is defined as the composition of operators on A. So we have

$$\left(u\partial^\alpha\right) \cdot \left(v\partial^\beta\right) = u \sum_{\gamma \in \dot{J}(\alpha)} \binom{\alpha}{\gamma} \partial^\gamma(v)\partial^{\alpha+\beta-\gamma}, \quad \forall u, v \in A;\ \alpha, \beta \in \dot{J}, \tag{2.1}$$

where

$$\dot{J}(\alpha) = \{\gamma \in \dot{J} \mid \gamma_i \leqslant \alpha_i,\ \forall i \in \dot{I}\},$$
$$\binom{\alpha}{\gamma} = \prod_{i \in \dot{I}} \binom{\alpha_i}{\gamma_i}, \quad \gamma \in \dot{J}(\alpha).$$

Denote $F_1 = A^D = \{x \in A \mid D(x) = 0\}$. It is well known from [P] that F_1 is a field extension of F if $A[D]$ is simple. From now on in this paper, we always use

$$\{d_i \mid i \in I\}$$

to denote an F_1-basis for $F_1 D$, and let

$$J = \{\alpha = (\alpha_i \mid i \in I) \mid \alpha_i \in \mathbb{Z}_+,\ \forall i \in I \text{ and } \alpha_i = 0 \text{ for all but a finite number of } i \in I\},$$

where $\alpha = (\alpha_i \mid i \in I)$. We again use the notation $d^\alpha = \prod_{i \in I} d^{\alpha_i}$, $\forall \alpha \in J$. Then

$$A[D] = \sum_{\alpha \in J} A d^\alpha.$$

For convenience of later use, we collect some related results from [Z2,Br].

Theorem 2.1. [Z2]

(a) *The center $Z(A[D])$ of $A[D]$ is F_1.*
(b) *The associative algebra $A[D]$ is simple if and only if A is D-simple, i.e., A has no nontrivial D-stable ideals.*
(c) *Suppose A is D-simple. Then*
  (1) *F_1 is a field extension of F,*
  (2) *$A[D]$ is a free left A-module with a basis $\{d^\alpha \mid \alpha \in J\}$,*
  (3) *$A[D]^L = [A[D]^L, A[D]^L]$, where $A[D]^L$ is the Lie algebra of $A[D]$.*

Now we recall the extended centroid from [M]. Let R be a prime ring (i.e., $aRb = 0$ implies $a = 0$ or $b = 0$). Define an equivalence relation on the set of all pairs (U, f) where U is a nonzero ideal of R and $f : U \to R$ is a right R-module mapping from U to R. Define $(U, f) \sim (V, g)$ if $f = g$ on some nonzero ideal $W \subseteq U \cap V$. The set Q of all equivalence classes forms a ring under the operations induced by addition and composition of representatives of equivalence classes. R embeds in Q as left multiplication on R. The center C of Q is a field containing the centroid of R. C is called the *extended centroid* of R. The C-algebra $C + RC$ is called the *central closure* of R. If R is a simple ring, then $Q = R$, and $C + RC = R$.

Lemma 2.2. [Br, Lemma 1] *Let R be a prime ring. Then the following are equivalent*:

(i) *R satisfies S_4*;
(ii) *R is commutative or R embeds in $M_2(F)$ for a field F*;
(iii) *R is algebraic of bounded degree 2 over C (i.e., for any $a \in R$ there exists a polynomial $x^2 + \alpha_1 x + \alpha_2 \in C[x]$ satisfied by a)*;
(iv) *R satisfies $[[x^2, y], [x, y]] = 0$.*

Theorem 2.3. [Br, Theorem 5] *Let R be a prime ring of characteristic not 2. Let d be a Lie derivation of R. If R does not satisfy S_4 then d is of the form $\delta + \tau$, where δ is a derivation of R into its central closure and τ is an additive mapping of R into C sending commutators to zero.*

Now we apply this theorem to our algebra $A[D]$.

Theorem 2.4. *If the Weyl type algebra $A[D]$ is simple, then* $\mathrm{Der}_F(A[D]) = \mathrm{Der}_F(A[D]^L)$.

Proof. It is clear that $R := A[D]$ is a prime ring. Since R is simple, we see that $A[D]$ is centrally closed, and C is the center of R. Let $0 \neq d \in D$. By Theorem 2.1 we have $d^2 + \alpha_1 d + \alpha_2 \neq 0$ for any $\alpha_1, \alpha_2 \in C = F_1$. Thus $A[D]$ does not satisfy any one of the conditions in Theorem 2.3, i.e., R does not satisfy S_4. From $[A[D], A[D]] = A[D]$, we know that τ is always 0 in Theorem 2.3, i.e., $\mathrm{Der}_F(A[D]^L) \subseteq \mathrm{Der}_F(A[D])$. It is also clear that $\mathrm{Der}_F(A[D]) \subseteq \mathrm{Der}_F(A[D]^L)$. Therefore $\mathrm{Der}_F(A[D]) = \mathrm{Der}_F(A[D]^L)$. □

3. Simple Weyl type algebras have nonzero-divisors

In this section, we prove our first main result which is about zero-divisors of the simple Weyl type algebra $A[D]$.

Theorem 3.1. *Let A be a unital commutative associative algebra of characteristic* 0*, D an abelian subalgebra of* $\mathrm{Der}_F(A)$*. Suppose A is D-simple. Then A is a domain, and $A[D]$ is a noncommutative domain.*

Proof. For any element $x \in A$, let $\mathrm{ann}(x) = \{y \in A \mid xy = 0\}$. Then $\mathrm{ann}(x^k) \subset \mathrm{ann}(x^{k+1})$, and $I(x) = \bigcup_{k=1}^{\infty} \mathrm{ann}(x^k)$ is a D-stable ideal of A. Since A is D-simple, then $I(x) = 0$ or $I(x) = A$. If $0 \neq x \in A$ is a zero-divisor, then $I(x) = A$, so there exists some $n \in \mathbb{N}$, so that $1 \in \mathrm{ann}(x^n)$, namely, $x^n = 0$. Thus all zero-divisors of A are nilpotent. It is easy to see that $Z = \{a \in A \mid a^n = 0 \text{ for some } n \in \mathbb{N}\}$ is an ideal (which is the set of all zero-divisors of A). For any nonzero $a \in Z$, there exists some n such that $a^n = 0$, and $a^{n-1} \neq 0$. Let $\partial \in D$, we have $\partial(a^n) = na^{n-1}\partial(a) = 0$. So $\mathrm{ann}(\partial(a)) \neq 0$. Therefore, $\partial(a) \in Z$, i.e., Z is a D-stable ideal. Since $Z \neq A$, thus $Z = 0$, that is, A is a domain.

From Theorem 2.1(c)(2), we can well define the degree $\deg xd^\alpha$ of the nonzero monomial xd^α to be α where $x \in A$. Consider a compatible total ordering on the group $\mathbb{Z}^J$, then we can define the degree of any element in R as usual. It is follows from Theorem 2.1(c)(2) and (2.1) that $\deg(XY) = \deg X + \deg Y$ fore all $X, Y \in R$. From this we see that $A[D]$ is a noncommutative domain. □

Remark. If $\mathrm{char}\, F = p \neq 0$, the statement in Theorem 3.1 does not hold. For example: let $F = \mathbb{Z}/p\mathbb{Z}$, $A = F[X]/(X^p)$, then A is $\partial/\partial X$-simple, while A is not a domain.

4. Extended Weyl type algebras

Throughout this section, we assume that $A[D]$ is simple and $\dim_{F_1} F_1 D = n < \infty$ and suppose $\{d_i \mid i = 1, 2, \ldots, n\}$ is an F_1-basis of $F_1 D$.

From Theorem 3.1 we know that A is a domain.

Lemma 4.1. *There exist $a_1, a_2, \ldots, a_n \in A$ such that* $\det(d_i(a_j))_{i,j=1}^n \neq 0$.

Proof. Let K be the fraction field of A and $M = \sum_{b \in A} K(d_1(b), \ldots, d_n(b)) \subseteq K^n$. Suppose $\dim_K M = m < n$. Choose a K-basis for M:

$$\big(d_1(b_1), \ldots, d_n(b_1)\big),\ \big(d_1(b_2), \ldots, d_n(b_2)\big),\ \ldots,\ \big(d_1(b_m), \ldots, d_n(b_m)\big).$$

For any $a \in A$, since $(d_1(a), d_2(a), \ldots, d_n(a)) \in M$, there exist $f_i(a) \in K$, such that $d_i(a) = f_1(a)d_i(b_1) + f_2(a)d_i(b_2) + \cdots + f_m(a)d_i(b_m)$, $\forall i = 1, 2, \ldots, n$. Choose $i_1, \ldots, i_m \in \{1, 2, \ldots, n\}$ such that $\det(d_{i_k}(b_j))_{k,j=1}^m \neq 0$ (we can rearrange the order of b_j if necessary). Denote by

B the $m \times m$ matrix $(d_{i_k}(b_j))_{k,j=1}^m$. We have $(f_1(a), f_2(a), \dots, f_m(a))^T = B^{-1}(d_{i_1}(a), d_{i_2}(a), \dots, d_{i_m}(a))^T$, where X^T denotes the transpose of the matrix X. Now for any $j \notin \{i_1, i_2, \dots, i_m\}$, we have

$$d_j(a) = \big(d_j(b_1), d_j(b_2), \dots, d_j(b_m)\big) B^{-1} \big(d_{i_1}(a), d_{i_2}(a), \dots, d_{i_m}(a)\big)^T, \quad \forall a \in A,$$

i.e.,

$$d_j = \big(d_j(b_1), d_j(b_2), \dots, d_j(b_m)\big) B^{-1} (d_{i_1}, d_{i_2}, \dots, d_{i_m})^T, \quad \forall a \in A.$$

By multiplying a suitable element of A to both sides, we get a nontrivial A-linear relation among $d_j, d_{i_1}, \dots, d_{i_m}$. So $d_j, d_{i_1}, \dots, d_{i_m}$ are A-linearly dependent, which contradicts Theorem 2.1(c). So we have proved the lemma. $\square$

Let $\partial \in \mathrm{Der}_F(A)$, and K be the fraction field of A. We can naturally extend ∂ to a derivation of K by defining $\partial(ab^{-1}) = \partial(a)b^{-1} - ab^{-2}\partial(b)$.

Lemma 4.2. *If $f \in K$ and $D(f) = 0$, then $f \in F_1$.*

Proof. We only need to prove that $f \in A$. Let $I(f) = \{a \in A \mid af \in A\}$. It is easy to check $I(f)$ is a D-stable ideal of A. So we have $I(f) = A$, and $1 \in I(f)$, which shows that $f \in A$. Therefore, $f \in F_1$. $\square$

For convenience, we define the F_1-subspace of A^n

$$S\big(A^n\big) = \big\{(f_1, f_2, \dots, f_n) \in A^n \mid d_i(f_j) = d_j(f_i),\ \forall i, j = 1, 2, \dots, n\big\}.$$

If $f^{(k)} = ({f_1}^{(k)}, {f_2}^{(k)}, \dots, {f_n}^{(k)}) \in S(A^n)$ for $k = 1, 2, \dots, s$, then we can extend D to commuting derivations of the commutative associative algebra $A[X_1, X_2, \dots, X_s] = A \otimes_F F[X_1, X_2, \dots, X_s]$ by defining $d_i(X_k) = {f_i}^{(k)}$. And we have the well-defined Weyl type algebra $A[X_1, X_2, \dots, X_s][D]$. We call it the *extended Weyl type algebra* of $A[D]$ by $\{f^{(k)}\}_{k=1}^s \subseteq S(A^n)$.

We define F_1-subspace of $S(A^n)$

$$V(A, D) = \big\{\big(d_1(a), \dots, d_n(a)\big) \mid a \in A\big\} \subseteq A^n.$$

Lemma 4.3. *Let $A[X][D]$ be the extended Weyl type algebra of $A[D]$ by the singleton $\{f\} \subseteq S(A^n)$.*

(a) *$A[X][D]$ is simple if and only if $f \notin V(A, D)$.*
(b) *If $A[X]$ is D-simple, then the center $Z(A[X][D]) = F_1$.*
(c) *If $A[X]$ is D-simple and $g \in A[X]$ satisfies $D(g) \subseteq A$, then $g \in A + F_1X$.*

Proof. We have assumed that A is D-simple.

(a) If $f = (d_1(a), d_2(a), \dots, d_n(a))$ for some $a \in A$, then the proper ideal of $A[D]$ generated by $X - a$ is D-stable. Thus $A[X][D]$ is not simple. Conversely, suppose $A[X][D]$ is not simple. Then $A[X]$ has a nontrivial D-stable ideal $\mathbf{b}$. Let $0 \neq b_0 \in \mathbf{b}$ have the lowest degree with respect to X. If $b_0 \in A$, then $\mathbf{b} \cap A$ is a nonzero D-stable ideal of A. We see $A = \mathbf{b} \cap A$, i.e., $A \subseteq \mathbf{b}$.

It follows that $\mathbf{b} = A[X]$, which contradicts the choice of $\mathbf{b}$. Thus $b_0 = a_0 + a_1X + \cdots + a_tX^t$ for some $a_i \in A$ with $a_t \neq 0$, and $t \geqslant 1$. Let

$$\mathbf{a} = \{c_t \mid c_0 + c_1X + \cdots + c_tX^t \in \mathbf{b} \text{ for some } c_i \in A,\ i = 0, 1, \ldots, t-1\}.$$

Since $a_t \in \mathbf{a}$ and $\mathbf{a}$ is a D-stable ideal of A, we deduce that $\mathbf{a} = A$, and $1 \in \mathbf{a}$. So $c = c_0 + c_1X + \cdots + c_{t-1}X^{t-1} + X^t \in \mathbf{b}$ for some $c_i \in A$. Since $d_i(c) \in \mathbf{b}$ and $\deg_X(d_i(c)) < t$, from the minimality of t we must have

$$d_i(c) = d_i(c_0) + \cdots + \bigl(d_i(c_{t-1}) + td_i(X)\bigr)X^{t-1} = 0,$$

for all $i = 1, 2, \ldots, n$. Thus $d_i(-c_{t-1}/t) = d_i(X) = f_i$ for all $i = 1, 2, \ldots, n$, i.e., $f \in V(A, D)$. This completes the proof of (a).

(b) We know that $A[X][D]$ is a simple Weyl type algebra, and $Z(A[X][D]) = \{g \in A[X] \mid D(g) = 0\}$ is a field extension of F. Note that the invertible elements of $A[X]$ are contained in A. From Theorem 2.1, we see that $Z(A[X][D]) = A \cap \{u \in A[X] \mid D(u) = 0\} = \{u \in A \mid D(u) = 0\} = F_1$.

(c) This is easy to check by using (a) and the last part of its proof. □

Let $M(f^{(1)}, f^{(2)}, \ldots, f^{(s)})$ be the F_1-vector space spanned by $f^{(1)}, f^{(2)}, \ldots, f^{(s)} \in S(A^n)$. Define the F_1-vector space

$$\bar{M}\bigl(f^{(1)}, f^{(2)}, \ldots, f^{(s)}\bigr) = \frac{M(f^{(1)}, f^{(2)}, \ldots, f^{(s)}) + V(A, D)}{V(A, D)}.$$

Corollary 4.4. *Let $A[X_1, X_2, \ldots, X_s][D]$ be the extended Weyl type algebra of $A[D]$ by $f^{(1)}, f^{(2)}, \ldots, f^{(s)} \in S(A^n)$.*

(a) *If $A[X_1, X_2, \ldots, X_s]$ is D-simple and $f \in A[X_1, X_2, \ldots, X_s]$ satisfies $D(f) \subset A$, then $f \in A + F_1X_1 + F_1X_2 + \cdots + F_1X_s$.*
(b) *If $A[X_1, X_2, \ldots, X_s]$ is D-simple, then $Z(A[X_1, X_2, \ldots, X_s][D]) = F_1$.*
(c) *$A[X_1, X_2, \ldots, X_s][D]$ is simple if and only if $\dim_{F_1} \bar{M}(f^{(1)}, f^{(2)}, \ldots, f^{(s)}) = s$.*

Proof. (a) If I is a nontrivial D-stable ideal of $A[X_1, X_2, \ldots, X_k]$, then the ideal $I[X_{k+1}, X_{k+2}, \ldots, X_s]$ is a nontrivial D-stable ideal of $A[X_1, X_2, \ldots, X_s]$. Thus $A[X_1, X_2, \ldots, X_k]$ is D-simple for $k = 1, 2, \ldots, s-1$, if $A[X_1, X_2, \ldots, X_s]$ is D-simple.

Applying Lemma 4.3(c) to $A[X_1, X_2, \ldots, X_{s-1}]$, we get $f \in A[X_1, X_2, \ldots, X_{s-1}] + F_1X_s$. Similarly $f \in A[X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_s] + F_1X_i$, $\forall i \in \{1, 2, \ldots, s\}$. Thus

$$f \in \bigcap_{i=1}^{s} \bigl(A[X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_s] + F_1X_i\bigr) = A + F_1X_1 + F_1X_2 + \cdots + F_1X_s,$$

(a) follows.

(b) Since invertible elements of $A[X_1, X_2, \ldots, X_s]$ are contained in A and $Z(A[X_1, X_2, \ldots, X_s][D])$ is a field contained in $A[X_1, X_2, \ldots, X_s]$, then by Theorem 2.1 we deduce that

$$Z\big(A[X_1,X_2,\dots,X_s][D]\big)=A\cap\big\{u\in A[X_1,X_2,\dots,X_s]\mid D(u)=0\big\}$$
$$=\big\{u\in A\mid D(u)=0\big\}=F_1.$$

(c) "$\Rightarrow$" Note that $A[x_1,x_2,\dots,x_k]$ is D-simple for $k=1,2,\dots,s$, and $V(A[x_1,x_2,\dots,x_k],D)\supset M(f^{(1)},f^{(2)},\dots,f^{(k)})+V(A,D)$. Repeatedly using Lemma 4.3(a), we easily obtain this direction.

"$\Leftarrow$" Suppose $\dim_{F_1}\bar{M}(f^{(k)})=s$. We shall show this direction by induction on s. If $s=1$, this is Lemma 4.3(a). Suppose the statement in (a) holds for $s-1$, i.e., $A[X_1,X_2,\dots,X_{s-1}][D]$ is simple. If $A[X_1,X_2,\dots,X_s][D]$ is not simple, then by Lemma 4.3(a) there exist an $a\in A[X_1,X_2,\dots,X_{s-1}]$ such that $d_i(a)=f_i^{(s)}$, where $f^{(s)}=(f_1^{(s)},f_2^{(s)},\dots,f_n^{(s)})$. Now by applying (a) to $A[X_1,X_2,\dots,X_{s-1}]$, we have $a\in A+F_1X_1+F_1X_2+\dots+F_1X_{s-1}$, i.e., $f^{(s)}\in M(f^{(1)},f^{(2)},\dots,f^{(s-1)})+V(A,D)$, which contradicts the assumption $\dim_{F_1}\bar{M}(f^{(k)})=s$. Therefore $A[X_1,X_2,\dots,X_s][D]$ is simple. □

5. Derivations of Weyl type algebras

We still assume that $A[D]$ is a simple Weyl type algebra and $\dim_{F_1}F_1D=n<\infty$, and suppose that $\{d_i\mid i=1,2,\dots,n\}$ is an F_1-basis of F_1D.

For any element $x=\sum_{\alpha\in J}u_\alpha d^\alpha\in A[D]$ where $u_\alpha\in A$, we define the degree of x as the maximal $|\alpha|$ with $u_\alpha\neq 0$, denoted by $\deg_D(x)$.

Lemma 5.1. *Suppose $A[D]$ is simple. Let $E\in \mathrm{Der}_F(A[D],A[X_1,X_2,\dots,X_s][D])$, where $A[X_1,X_2,\dots,X_s][D]$ is a simple extended Weyl type algebra of $A[D]$. Let $a_1,a_2,\dots,a_n$ be as in Lemma* 4.1 *and $m=\max\{\deg_D(E(a_i))\mid i=1,2,\dots,n\}$. Then $\deg_D(E(a))\leqslant m$ for all $a\in A$.*

Proof. Suppose there exists $b\in A$ such that $\deg_D(E(b))=N>m$. Let

$$E(b)=\sum_\alpha E_\alpha(b)d^\alpha,\quad \forall b\in A,$$

where $E_\alpha(b)\in A[x_1,\dots,x_s]$. From $[E(a_i),b]+[a_i,E(b)]=0$ (considering the homogeneous term of degree $N-1$), we deduce that

$$\sum_{\beta\in\mathbb{Z}_+^n:\,|\beta|=N-1}\left(\sum_{k=1}^n E_{\beta+\varepsilon_k}(b)(\beta_k+1)d_k(a_i)\right)d^\beta=0,\quad \forall i=1,2,\dots,n.$$

From Theorem 2.1, we deduce that

$$\sum_{k=1}^n E_{\beta+\varepsilon_k}(b)(\beta_k+1)d_k(a_i)=0,\quad \forall i=1,2,\dots,n,$$

for all $\beta\in\mathbb{Z}_+^n$ with $|\beta|=N-1$. Since $\det(d_i(a_j))\neq 0$ we deduce that $E_{\beta+\varepsilon_k}(b)=0$ for all $|\beta|=N-1$. This is a contradiction to the degree of $E(b)$. □

For any $E \in \mathrm{Der}_F(A)$, if $[E, D] \subseteq AD$ as derivations of A, then we can naturally extend it to a derivation $\bar{E}$ of the associative algebra $A[D]$ by defining $\bar{E}(d) = [E, d]$ for all $d \in D$. Set

$$\mathrm{Ext}\big(A[D]\big) = \big\{\bar{E} \mid E \in \mathrm{Der}_F(A) \text{ with } [E, D] \subseteq AD\big\}. \tag{5.1}$$

We use Ext to remind us that these derivations are extensions from some derivations of A.

For any $f \in S(A^n)$, let $A[X][D]$ be the extended Weyl type algebras of $A[D]$ by the singleton $\{f\}$. Then $\mathrm{ad}(X) \in \mathrm{Der}(A[X][D])$, and $\partial_f := \mathrm{ad}(X)|_{A[D]}$ is a derivation of $A[D]$. Set

$$\mathrm{Res}\big(A[D]\big) = \big\{\partial_f \mid f \in S\big(A^n\big)\big\}. \tag{5.2}$$

It is easy to see that $\mathrm{Res}(A[D])$ is abelian, and acts trivially on A. We use Res since it can remind us that these derivations are restrictions of some derivations from $A[X][D]$.

Note that $\partial_f(A) = 0$ for any $f \in S(A^n)$. Thus by now we know three classes of derivations on $A[D]$: $\mathrm{Ext}(A[D])$, $\mathrm{Res}(A[D])$, and the inner derivations $\mathrm{ad}(A[D])$.

Lemma 5.2. *If $E \in \mathrm{Der}_F(A[D])$, $E(D) \subseteq A + AD$ and $E(A) \subseteq A$, then $E \in \mathrm{Ext}(A[D]) \oplus \mathrm{Res}(A[D])$.*

Proof. It is clear that the sum $\mathrm{Ext}(A[D]) + \mathrm{Res}(A[D])$ is direct. Let $E(d_i) = a_i + \sum_{k=1}^n b_k^{(i)} d_k$. From $[d_i, d_j] = 0$, we deduce that $(a_1, \ldots, a_n) \in S(A^n)$. Then there exists a derivation $E_2 \in \mathrm{Res}(A[D])$ with $E_2(d_i) = a_i$ for all i. Letting $E_1 = E - E_2$, we know that

$$E_1(d_i) \in AD, \qquad E_1(A) \subseteq A.$$

Since $E_1([d_i, a]) = [E_1(d_i), a] + [d_i, E_1(a)]$ for all $a \in A$, we have, as operators on A,

$$\begin{aligned}\big(E_1(d_i)\big)(a) &= \big[E_1(d_i), a\big] = E_1\big([d_i, a]\big) - \big[d_i, E_1(a)\big] \\ &= \big[E_1, [d_i, a]\big] - \big[d_i, [E_1, a]\big] = \big[[E_1, d_i], a\big] = [E_1, d_i](a).\end{aligned}$$

Thus $E_1(d_i) = [E_1, d_i]$ as operators on A. Let $Y = E_1|_A \in \mathrm{Der}_F(A)$. From $[Y, d_i](a) = [E_1, d_i](a) = (E_1(d_i))(a)$, we see that $E_1(d_i) = [Y, d_i]$ as operators of A. Then $E_1 = \bar{Y} \in \mathrm{Ext}(A[D])$. $\square$

Now we are ready to present our second main result of this paper.

Theorem 5.3. *Let A be a unital commutative associative algebra over a field F of characteristic 0, D a nonzero abelian subalgebra of $\mathrm{Der}_F(A)$. Suppose A is D-simple and $\{d_1, d_2, \ldots, d_n\}$ is an F_1-basis for $F_1 D$. Then*

$$\begin{aligned}\mathrm{Der}_F\big(A[D]^L\big) &= \mathrm{Der}_F\big(A[D]\big) = \mathrm{ad}\big(A[D]\big) + \mathrm{Ext}\big(A[D]\big) + \mathrm{Res}\big(A[D]\big) \\ &= \mathrm{ad}\Big(\sum_{\alpha \in \mathbb{Z}_+^n:\ |\alpha| > 1} Ad^\alpha\Big) \oplus \mathrm{Ext}\big(A[D]\big) \oplus \mathrm{Res}\big(A[D]\big).\end{aligned}$$

Proof. Fix $E \in \mathrm{Der}_F(A[D])$, and let

$$E(d_i) = \sum_{\alpha\in\mathbb{Z}_+^n} f_i^{(\alpha)} d^\alpha, \quad \text{where } f_i^{(\alpha)} \in A. \tag{5.3}$$

Denote $f^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)}, \ldots, f_n^{(\alpha)}) \in A^n$. Applying E to $[d_i, d_j] = 0$, we obtain that $[E(d_i), d_j] = [E(d_j), d_i]$, to give $d_i(f_j^{(\alpha)}) = d_j(f_i^{(\alpha)})$ for any $\alpha \in J$ and $i, j = 1, 2, \ldots, n$. So $f^{(\alpha)} \in S(A^n)$. Since there are only finitely many nonzero $f^{(\alpha)}$, then $s := \dim_{F_1} \bar{M}(f^{(\alpha)} \mid \alpha \in \mathbb{Z}_+^n) < \infty$. Now choose $f^{(\alpha_1)}, f^{(\alpha_2)}, \ldots, f^{(\alpha_s)} \in \{f^{(\alpha)} \mid \alpha \in \mathbb{Z}_+^n\}$ such that the images of $\{f^{(\alpha_i)}\}_{i=1}^s$ form an F_1-basis of $\bar{M}(f^{(\alpha)} \mid \alpha \in \mathbb{Z}_+^n)$. We have the extended Weyl type algebra $\bar{A}[D] = A[X_1, X_2, \ldots, X_s][D]$ by $\{f^{(\alpha_i)}\}_{i=1}^s$. By Corollary 4.4, we know that $\bar{A}[D]$ is simple, and $Z(\bar{A}[D]) = F_1$ $(= Z(A[D]))$. Note that $d_j(X_i) = f_j^{(\alpha_i)}$. For any α, write $f^{(\alpha)} = \sum_{i=1}^s h_i^{(\alpha)} f^{(\alpha_i)} + (d_1(a^\alpha), d_2(a^{(\alpha)}), \ldots, d_n(a^{(\alpha)}))$ for some $h_i^\alpha \in F_1$, $a^{(\alpha)} \in A$. Then $(d_1(h_\alpha), \ldots, d_n(h_\alpha)) = f^{(\alpha)}$, where $h_\alpha = a^{(\alpha)} + \sum_{i=1}^s h_i^{(\alpha)} X_i$. Let

$$h = \sum_{\alpha\in\mathbb{Z}_+^n} h_\alpha d^\alpha. \tag{5.4}$$

Denote $E' = E + \operatorname{ad} h \in \mathrm{Der}_F(A[D], \bar{A}[D])$. Then

$$E'(d_i) = 0, \quad \forall i = 1, 2, \ldots, n. \tag{5.5}$$

Let $a_1, a_2, \ldots, a_n \in A$ be as in Lemma 4.1, and $m = \max\{\deg_D E'(a_i) \mid i = 1, 2, \ldots, n\}$.

Claim 1. *If $m \geqslant 1$, then there exists $Y \in \sum_{|\alpha|=m+1} F_1 d^\alpha$ such that $\deg_D(E' - \operatorname{ad} Y)(a_i) < m$, for $i = 1, 2, \ldots, n$.*

Let

$$E'(b) = \sum_{\alpha\in\mathbb{Z}_+^n} E_\alpha(b) d^\alpha, \quad \forall b \in A, \tag{5.6}$$

where $E_\alpha(b) \in \bar{A}$, and suppose $m \geqslant 1$. By Lemma 5.1, we see that

$$E_\alpha(b) = 0 \quad \text{for any } b \in A,\ \alpha \in \mathbb{Z}_+^n \text{ with } |\alpha| > m.$$

From $[a, b] = 0$ for any $a, b \in A$, We have $[E'(a), b] + [a, E'(b)] = 0$, to give

$$\sum_{\beta\in\mathbb{Z}_+^n:\ |\beta|=m-1} \sum_{k=1}^n (\beta_k + 1) E_{\beta+\varepsilon_k}(a) d_k(b) d^\beta = \sum_{\beta\in\mathbb{Z}_+^n:\ |\beta|=m-1} \sum_{k=1}^n (\beta_k + 1) E_{\beta+\varepsilon_k}(b) d_k(a) d^\beta,$$

where $\varepsilon_k = (\delta_{1,k}, \ldots, \delta_{n,k}) \in \mathbb{Z}_+^n$. From Theorem 2.1, we deduce that

$$\sum_{k=1}^{n}(\beta_k+1)E_{\beta+\varepsilon_k}(a)d_k(b)=\sum_{k=1}^{n}(\beta_k+1)E_{\beta+\varepsilon_k}(b)d_k(a),\quad \forall a,b\in A, \tag{5.7}$$

for all $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{Z}_+^n$ with $|\beta|=m-1$. Letting $a=a_1,a_2,\dots,a_n$ respectively, we have

$$\begin{aligned}&\big((\beta_k+1)E_{\beta+\varepsilon_k}(a_j)\big)_{j,k=1}^{n}\big(d_1(b),\dots,d_n(b)\big)^T\\ &\quad=\big((\beta_k+1)d_k(a_j)\big)_{j,k=1}^{n}\big(E_{\beta+\varepsilon_1}(b),E_{\beta+\varepsilon_2}(b),\dots,E_{\beta+\varepsilon_n}(b)\big)^T.\end{aligned}$$

Since $\det(d_k(a_j))\neq 0$, then

$$E_{\beta+\varepsilon_i}(b)=\sum_{t=1}^{n}f_{\beta+\varepsilon_i}^{(t)}d_t(b),\quad \forall b\in A, \tag{5.8}$$

where $f_{\beta+\varepsilon_i}^{(t)}\in\bar{K}$, the fraction field of $\bar{A}=A[X_1,X_2,\dots,X_s]$. Since $E'(d_j)=0$, we have $E'([d_j,b])=[d_j,E'(b)]$, i.e., $E_{\beta+\varepsilon_i}(d_j(b))=d_j(E_{\beta+\varepsilon_i}(b))$. Together with (5.8) we obtain that

$$\sum_{t=1}^{n}d_j\big(f_{\beta+\varepsilon_i}^{(t)}\big)d_t(b)=0,\quad \forall b\in A,\ i,j\in\{1,2,\dots,n\}.$$

By taking b as a_j for $1\leqslant j\leqslant n$ in Lemma 4.1, we deduce that $d_j(f_{\beta+\varepsilon_i}^{(t)})=0$ for all $1\leqslant i,j\leqslant n$. Now applying Lemma 4.2 to $\bar{A}$, we have $f_{\beta+\varepsilon_i}^{(t)}\in F_1$ for all i and t. Applying (5.8) to (5.7) we deduce that

$$\sum_{j=1}^{n}\sum_{t=1}^{n}f_{\beta+\varepsilon_j}^{(t)}(\beta_j+1)d_t(a)d_j(b)=\sum_{j=1}^{n}\sum_{k=1}^{n}f_{\beta+\varepsilon_k}^{(j)}(\beta_k+1)d_k(a)d_j(b),\quad \forall a,b\in A,$$

simplifying it to give

$$\sum_{t=1}^{n}f_{\beta+\varepsilon_j}^{(t)}(\beta_j+1)d_t(a)=\sum_{k=1}^{n}f_{\beta+\varepsilon_k}^{(j)}(\beta_k+1)d_k(a),\quad \forall a\in A.$$

Thus $f_{\beta+\varepsilon_j}^{(l)}(\beta_j+1)=f_{\beta+\varepsilon_l}^{(j)}(\beta_l+1)$, i.e.,

$$\frac{f_{\beta+\varepsilon_j}^{(l)}}{\beta_l+1}=\frac{f_{\beta+\varepsilon_l}^{(j)}}{\beta_j+1},\quad \forall 1\leqslant j,l\leqslant n,$$

for all $\beta\in\mathbb{Z}_+^n$ with $|\beta|=m-1$. From this formula it is not difficult to verify that

$$c_{\beta+\varepsilon_j+\varepsilon_k}=\frac{f_{\beta+\varepsilon_j}^{(k)}}{\beta_k+\delta_{j,k}+1}\in F_1$$

is well defined for all $\beta \in \mathbb{Z}_+^n$ with $|\beta| = m - 1$. Let $Y = \sum_{|\alpha|=m+1} c_\alpha d^\alpha$. From (5.8), the coefficient of $d^{\beta+\varepsilon_j}$ for $|\beta| = m - 1$ in $(E' - \operatorname{ad} Y)(a_k)$ is

$$E_{\beta+\varepsilon_j}(a_k) - \sum_{t=1}^n c_{\beta+\varepsilon_j+\varepsilon_t}(\beta_t + \delta_{t,j} + 1)d_t(a_k) = \sum_{t=1}^n f_{\beta+\varepsilon_j}^{(t)} d_t(a_k) - \sum_{t=1}^n f_{\beta+\varepsilon_j}^{(t)} d_t(a_k) = 0.$$

Thus $\deg_D(E' - \operatorname{ad} Y)(a_i) < m$ as desired. Claim 1 follows.

Now from Claim 1, there exists an $h' \in F_1[D]$, such that $(E' - \operatorname{ad} h')(a_i) \in \bar{A}$. Denote $E'' = E' - \operatorname{ad} h'$. We still have $E''(D) = 0$. By Lemma 5.1, we have

$$E''(A) \subseteq \bar{A}. \tag{5.9}$$

Since $E = E'' - \operatorname{ad} h + \operatorname{ad} h'$ and $E(A) \subseteq A[D]$, we have $\operatorname{ad} h(A) \subseteq \bar{A} + A[D]$. Combining this with (5.4) we deduce that $h \in A[D] + \sum_{i=1}^n \sum_{j=1}^s (F_1 X_j d_i + F_1 X_j)$. Let $h = h'' + \sum_{i=1}^n \sum_{j=1}^s (h_{i,j} X_j d_i + h_j X_j)$, where $h'' \in A[D]$ and $h_{i,j}, h_j \in F_1$. Denote

$$E''' = E - \operatorname{ad} h' + \operatorname{ad} h'' = E'' - \operatorname{ad}(h - h'') \in \operatorname{Der}_F\big(A[D]\big). \tag{5.10}$$

Using the fact that $E''(D) = 0$, we deduce that

$$E'''(d_i) = -\left[\sum_{i=1}^n \sum_{j=1}^s (h_{i,j} X_j d_i + h_j X_j), d_i\right] \in A + AD. \tag{5.11}$$

Using (5.9) and (5.10), we deduce that $E'''(a) = E''(a) - [\sum_{i=1}^n \sum_{j=1}^s h_{i,j} X_j d_i, a] \in \bar{A}$ for all $a \in A$. Combining this with (5.10) we deduce that

$$E'''(A) \subseteq \bar{A} \cap A[D] = A. \tag{5.12}$$

From (5.11), (5.12) and using Lemma 5.2, we see that $E''' \in \operatorname{Ext}(A[D]) + \operatorname{Res}(A[D])$. Note that $E = E''' + \operatorname{ad}(h' - h'')$ and $h', h'' \in A[D]$. Thus

$$\operatorname{Der}_F\big(A[D]\big) = \operatorname{ad} A[D] + \operatorname{Ext}\big(A[D]\big) + \operatorname{Res}\big(A[D]\big).$$

The direct sum $\operatorname{ad}(\sum_{\alpha \in \mathbb{Z}_+^n:\, |\alpha|>1} A d^\alpha) \oplus \operatorname{Ext}(A[D]) \oplus \operatorname{Res}(A[D])$ is clear. This completes the proof of this theorem. $\square$

6. Some applications

Theorem 5.3 tells us that determining $\operatorname{Der}(A[D])$ is equivalent to determining $\operatorname{Der}_F(A)$ and the set $S(A^n)$. This is very easy in some cases.

Example 1. Now we are going to use our theorem to recover the main theorem in [S]. Here we do not like to repeat all the definitions in [S] since they are too complicated. From [S], we know that $A = F[\Gamma \times \mathbb{Z}_+^{l_1+l_2}]$—the group algebra, $D = D_1 + D_2 + D_3$, and $\mathcal{W} = \mathcal{W}(l_1, l_2, l_3, \Gamma)$. It is easy to check that

$$\mathrm{Der}(A) = AD + \mathrm{span}\big\{uf \mid u \in A,\ f \in \hom_{\mathbb{Z}}(\Gamma, F)\big\}.$$

From $[\partial, uf] = \partial(u)f \in AD$ we deduce that $u \in F$ or $f \in D$. So

$$\mathrm{Ext}\big(A[D]\big) = \mathrm{ad}(AD) + \hom(\Gamma, F).$$

Let

$$(f_1, \ldots, f_l) = \Bigg(\sum_{\alpha\in\Gamma,\, i\in\mathbb{Z}_+^{l_1+l_2}} a_{\alpha,i}^{(1)} x^{\alpha,i}, \ldots, \sum_{\alpha\in\Gamma,\, i\in\mathbb{Z}_+^{l_1+l_2}} a_{\alpha,i}^{(l)} x^{\alpha,i} \Bigg) \in S\big(A^l\big),$$

where $a_{\alpha,i}^{(j)} \in F$. By computing $d_k(f_l) = d_l(f_k)$, we deduce that there exists an $u \in A$ such that $d_i(u) - f_i \in F$ for all $1 \leqslant i \leqslant l$. Thus

$$\mathrm{Res}\big(A[D]\big) = \mathrm{ad}(A) \oplus \Delta,$$

where $\Delta = \mathrm{span}\{\sigma_i \mid i = l_1 + l_2 + 1, \ldots, l\}$, $\sigma_i(A) = 0$, $\sigma_i(\partial_j) = \delta_{i,j}$ for $i = l_1 + l_2 + 1, \ldots, l$, $j = 1, \ldots, l$. Thus $\mathrm{Der}(A[D])$ is completely determined.

Example 2. Let $A = F(x, y)$, the fraction field of the polynomial ring $F[x, y]$, $\partial_x = x\frac{\partial}{\partial x}$, and $D = F\partial_x$. Then A is D-simple, $\mathrm{Der}_F(A) = A\partial_x + A\partial_y$ where $\partial_y = y\frac{\partial}{\partial y}$, $F_1 = F(y)$. This Weyl type algebra $A[D]$ does not belong to the case studied in [S]. It is clear that $\{\partial \in \mathrm{Der}_F(A) \mid [\partial, \partial_x] \subseteq A\partial_x\} = F(y)\partial_y + A\partial_x$. So

$$\mathrm{Ext}\big(A[D]\big) = \mathrm{ad}\big(F(y)\partial_y\big) + \mathrm{ad}(A\partial_x).$$

It is easy to see that $S(A^1) = A$. For any $f \in A$, define $\partial_f \in \mathrm{Der}_F(A[D])$ by $\partial_f(A) = 0$, and $\partial_f(\partial_x) = f$. Then

$$\mathrm{Res}\big(A[D]\big) = \{\partial_f \mid f \in A\}.$$

Thus $\mathrm{Der}(A[D])$ is completely determined by applying Theorem 5.3.

Example 3. Let $A = \mathbb{R}[x, p^{-1}]$, where $p = x^n + \sum_{i=0}^{n-1} a_i x^i$, $a_i \in \mathbb{R}$, $n \geqslant 1$. Let $D = \mathbb{R}\frac{\partial}{\partial x}$. Then A is D-simple, and $\mathrm{Der}_{\mathbb{R}}(A) = AD$. Hence we have $\mathrm{Ext}(A[D]) = \mathrm{ad}(AD)$. Let $p = \prod_{i=1}^k p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$, where $p_i, i = 1, 2, \ldots, k$, are co-prime irreducible polynomials of degree 1 or 2. For any $f \in A$, define $\partial_f \in \mathrm{Der}_F(A[D])$ by $\partial_f(A) = 0$, and $\partial_f(\frac{\partial}{\partial x}) = f$, then we have $\mathrm{Res}(A[D]) = \{\partial_f \mid f \in A\}$. Set

$$G = \left\{ \frac{1}{p_i} \;\middle|\; i = 1, 2, \ldots, k \right\} \cup \left\{ \frac{x}{p_i} \;\middle|\; \deg_x(p_i) = 2;\ i = 1, 2, \ldots, k \right\}.$$

From a direct computation which we leave for the readers, we deduce that

$$\operatorname{Res}\big(A[D]\big) = \bigoplus_{g \in G} \mathbb{R}\partial_g \oplus \operatorname{ad}(A).$$

Thus by applying Theorem 5.3 we know that

$$\operatorname{Der}\big(A[D]\big) = \operatorname{ad}\big(A[D]\big) \bigoplus_{g \in G} \mathbb{R}\partial_g.$$

Example 4. We simply replace $\mathbb{R}$ by any algebraically closed field of characteristic 0 in Example 3. Then p_i, $i = 1, 2, \dots, k$, are only co-prime irreducible polynomials of degree 1. Then we have $\operatorname{Res}(A[D]) = \{\partial_f \mid f \in A\}$. Set

$$T = \left\{ \frac{1}{p_i} \;\middle|\; i = 1, 2, \dots, k \right\}.$$

It is clear that

$$\operatorname{Res}\big(A[D]\big) = \bigoplus_{g \in T} \mathbb{R}\partial_g \oplus \operatorname{ad}(A).$$

Thus $\operatorname{Der}(A[D])$ is completely determined by applying Theorem 5.3. It is clear that $\dim(\operatorname{Der}(A[D])/\operatorname{ad}(A[D])) = k$, the degree of p.

The last two examples are very interesting since we can use the commuting outer derivations (all the outer derivations constructed above are commutative) to obtain Weyl type algebras in the manner as [Z3]. We think this will be an interesting project to study.

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